functions in the solution of the problem of $G_{6,2}$ as has been done in the case of the sextic.

The developments sketched above suggest a number of inquiries which promise results of interest in various directions. Also some of the problems considered such as that of the invariants of P_n^k are worthy of closer study than has been given them in the account here reported.

[1] This investigation has been pursued under the auspices of the Carnegie Institution of Washington, D. C.

[2] For references see Study, *Math. Ann., Leipzig*, **60**, 348 (footnote).

[3] E. H. Moore, *Amer. J. Math.*, **22**, 279 (1900).

[4] *Trans. Amer. Math. Soc.*, **9**, 396 (1908), and **12**, 311 (1911). The relation of these papers to the earlier work of Klein and others is there set forth at length.

[5] S. Kantor has used this device for an S_2 in his crowned memoir: *Premiers Fondements pour une Théorie des Transformations Périodiques Univoques*, Naples (De Rubertis), 1891.

[6] This will appear as a joint paper by Mr. C. P. Sousley and A. B. Coble.

[7] Letter to Hermite, *J. Math., Paris*, Ser. 4, **4**, 169. See also *Math. Ann., Leipzig*, Witting, **29**, 167; Maschke, **33**, 317, **36**, 190; and Burkhardt, **38**, 161.

THE STRAIGHT LINES ON MODULAR CUBIC SURFACES

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1. In ordinary space a cubic surface without singular points contains exactly 27 straight lines, of which 27, 15, 7, or 3 are real; there are 45 sets of three coplanar lines, the three of no set being concurrent. In modular space, in which the coördinates of points and the coefficients of the equations of lines or surfaces are integers or Galois imaginaries taken modulo 2, it is interesting to notice that *three coplanar lines on a cubic surface may be concurrent* (§2). A point with integral coördinates is called real. A line or surface is called real if the coefficients of its equations are integers. In space with modulus 2, the number of real straight lines on a cubic surface without singular points is 15, 9, 5, 3, 2, 1, or 0.

We shall give here an elementary, self-contained, investigation of some of the most interesting cubic surfaces modulo 2. A complete classification of all such surfaces under real linear transformation will appear in the *Annals of Mathematics*, but without the present investigation of the configuration of their lines.

2. Every real point of space modulo 2 is on the surface[1]

$$xy(x+y) = zw(z+w). \tag{1}$$

since each member is an even integer when x, y, z, w are integers.

Theorem 1. *This surface with exactly 15 real points contains exactly 15 real straight lines. There are exactly 15 sets of three coplanar real lines on the surface, the three of each set being concurrent, while their plane is the tangent to the surface at the point of concurrence.*

The surface has the automorphs[2] (i.e., transformations carrying the surface into itself)

$$(zw);\ \pi = (xz)(yw);\ \beta\colon z^1 = z + w;$$
$$\gamma\colon x^1 = x,\ y^1 = y + w,\ z^1 = z + x,\ w^1 = x + z + w.$$

The fifteen real points of space

1 = (1000)	5 = (1100)	9 = (0101)	13 = (1011)
2 = (0100)	6 = (1010)	10 = (0011)	14 = (0111)
3 = (0010)	7 = (1001)	11 = (1110)	15 = (1111)
4 = (0001)	8 = (0110)	12 = (1101)	

are permuted transitively by these automorphs, since

$$(zw)\colon (3\ 4)\ (6\ 7)\ (8\ 9)\ (11\ 12),$$
$$\beta\colon (4\ 10)\ (7\ 13)\ (9\ 14)\ (12\ 15),$$
$$\pi\colon (1\ 3)\ (2\ 4)\ (5\ 10)\ (7\ 8)\ (11\ 13)\ (12\ 14),$$
$$\gamma\colon (1\ 13\ 12\ 6)\ (3\ 10\ 8\ 14)\ (4\ 9)\ (5\ 15\ 7\ 11).$$

The tangents to (1) at the fifteen real points give all of the fifteen real planes in space, since the tangent at $(XYZW)$ is

$$Y^2x + X^2y + W^2z + Z^2w = 0.$$

Hence these planes are permuted transitively by the real linear automorphs of the surface. The plane $x = 0$, tangent at 2, contains the concurrent lines

$$A\colon x = z = 0, \qquad B\colon x = w = 0, \qquad C\colon x = 0,\ z = w,$$

evidently on the surface. Hence each of the fifteen real planes contains three (concurrent) lines lying on the surface (1). The only real planes through line A are evidently $x = 0$, $z = 0$, and $x = z$. Hence the number of real lines on (1) is 15.3/3. We have therefore proved Theorem 1.

Since (zw) and β replace A by B and C and since the fifteen real planes are permuted transitively by the automorphs, the fifteen real lines on the surface are permuted transitively by the real linear automorphs.

We next make use of the imaginary automorph I which leaves z and w unaltered, but multiplies x and y by t, where $t^2+t+1 = 0$. It replaces the line $w = x$, $y = z$ on (1) by E_{t0} $(w = tx,\ z = ty)$, where

$$E_{ab}\colon w = ax,\ y = bx + a^2z;\ D_{abc}\colon z = ax + bw,\ y = cx + a^2w.$$

The twelve imaginary lines in which b and c are integers, while $a^2 + a + 1 = 0$, are seen to be permuted transitively by (zw), π, β. We now have $15 + 12$ lines on (1). As in the algebraic theory, there are only 27 lines on a cubic surface without singular points.[3] Hence the 27 lines are permuted transitively by I and the real automorphs. To prove Theorem 2 below, it therefore suffices to take A as one of the three coplanar lines. The only planes through A are $z = 0$ and $x = kz$. The former cuts the surface in the lines A, $z = y = 0$, and $z = 0$, $x = y$, which concur at 4. The latter cuts the surface in the lines A and

$$x = kz, \; ky(y + kz) + w(z + w) = 0.$$

This quadratic function has the factor $w + \alpha y + \beta z$ if and only if $\beta^2 = \beta$, $k = \alpha^2$, $\alpha = k^2$. Hence A meets just 10 lines on the surface, the above two and

$$L_{k\beta}: \; x = kz, w = k^2 y + \beta z \qquad (\beta = 0, 1; \; k^4 = k).$$

The only coplanar sets of three lines, one of which is A, are the above set of three real and the four sets A, L_{k0}, L_{k1}, which meet at $(010k^2)$. In all there are $27.5/3 = 45$ sets of coplanar lines.

Theorem 2. *The 27 straight lines on (1) form 45 sets of three coplanar lines. The three of each set concur and their plane is the tangent to the surface at the point of concurrence.*

3. There are several types of real cubic surfaces the numbers of whose real points and real lines differ from those of (1), but are derivable from (1) by imaginary linear transformation and hence have 45 sets of three coplanar concurrent straight lines.

If in (1) we replace z by $ez + e^2w$ and w by $e^2z + e^4w$, where $e^3 + e + 1 = 0$, and hence replace $z + w$ by $e^4z + ew$, we evidently get

$$xy(x + y) = z^3 + zw^2 + w^3. \tag{2}$$

For integral values of the variables, the left number is zero modulo 2, whence $z \equiv w \equiv 0$. Thus 1, 2, 5 are the only real points on (2); it contains no real line.

If in (1) we make also the like replacement of x, y, we get

$$x^3 + xy^2 + y^3 = z^3 + zw^2 + w^3. \tag{3}$$

It contains just nine real points and only three real lines: $z = x$, $y = w$; $z = y = x + w$; $w = x = y + z$, no two of which intersect.

If in (1) we replace[4] z by $z + bw$ and w by $z + b^2w$, where $b^2 + b + 1 = 0$, we get

$$xy(x + y) = w(z^2 + zw + w^2). \tag{4}$$

The only real points on it are evidently the seven in the plane $w = 0$. Hence the only real lines on it are the three having $w = 0$.

If in (2) we replace x by $x + by$ and y by $x' + b^2y$, where $b^2 + b + 1 = 0$, we get

$$y(x^2 + xy + y^2) = z^3 + zw^2 + w^3. \tag{5}$$

Replacing y by $y + z + w$, z by $z + y$, w by $w + y$, we get a surface whose real points are those on the cone $yz + yw + zw = 0$ and hence are 1, 2, 3, 4, 5, 6, 7. Thus the real points are those on the lines $y = z = 0$, $y = w = 0$, $z = w = 0$, meeting at 1.

If in (1), written in Capitals, we set

$$X = x + y,\ Y = bx + b^2y + (a^2 + 1)w,\ Z = ax + (a + 1)y + z,$$
$$W = w,\ b^2 + b + 1 \equiv 0, a^4 + a + 1 \equiv 0,$$

we get the real surface[5]

$$x^3 + y^3 + (x^2 + z^2)w + (y + z)w^2 = 0, \tag{6}$$

which has only eleven real points and only five real lines

$$w = 0, y = x; \qquad z = x + tw, y = x; \quad z = tw, y = x + w \quad (t = 0, 1).$$

4. A surface without singular points and not having every set of three coplanar lines concurrent, as was the case with all the preceding surfaces, is

$$xy(x + y) = w(w + z)(y + z). \tag{7}$$

It contains thirteen real points, 9 and 12 alone being not on it. We obtain a straight line on (7) by equating to zero one factor of each member. The resulting nine lines will be shown later to be the only real lines on (7). The three in $x = 0$ are not concurrent, likewise the three in $x = y$, while the three in $y = 0$ meet at 1. We obtain a redistribution of these nine lines by beginning with a factor of the second member of (7): the three lines in $w = 0$ meet at 3, the three in $w = z$ meet at 10, the three in $y = z$ meet at 4. There is no plane other than these six which contains three coplaner lines from this set of nine real lines.

Evident automorphs[6] of (7) are

$$(A)\ z^1 = z + y + w; \qquad (B)\ w^1 = w + z; \qquad (C)\ x^1 = x + y.$$

Evident imaginary lines on (7) are

$$(H_a)\quad w = ax, z = ay \qquad (a^3 + a^2 + 1 = 0).$$

Applying the real automorphs to H_a, we get the new lines

$$(D_a)\quad z = ax + w, y = a^2w;$$
$$(E_a)\quad z = ax + w, y = x + (a + a^2)w;$$

$$(F_a)\quad z = ax + a^2w,\ y = a^2w;$$
$$(L_a)\quad w = ax,\ y = (a^2+1)x + a^2z;$$
$$(J_a)\quad z = (a+1)x + (a^2+a)w,\ y = x + (a^2+a)w.$$

Those with the same subscript a are permuted as follows:

$$A\colon (DF)(LH)(EJ),\quad B\colon (DL)(HE)(FJ),\quad C\colon (DF)(LJ)(EH).$$

Since the 6.3 imaginary lines and the earlier 9 lines give the 27 possible lines, we now have all the lines on (7), a fact checked otherwise.

The automorphs were seen to replace the three lines H_a by all of the 18 imaginary lines. Hence in studying sets of three coplanar lines at least one of which is imaginary, H_a may be taken as one line. The following are seen to give all such sets:

H_a, F_a and $x = y$, $w = z$, in $w + z = a(x + y)$;
H_a, E_a, $y = z = 0$, in $z = ay$;
H_a, D_{a^2}, J_{a^2+a+1}, in $a(w + ax) = ay + z$;
H_a, L_a, $x = w = 0$, in $w = ax$;
H_a, J_{a^2}, D_{a^2+a+1}, in $(a^2+1)(w + ax) = ay + z$.

Those in the second set alone are concurrent, meeting at $(100a)$. Hence there are $18.2/3 = 12$ sets of three imaginary non-concurrent coplanar lines and $18.3/2 = 27$ sets with two imaginary and one real coplanar lines, in 9 of which sets the lines concur (since A, B, C permute the pairs HE, LJ, DF).

Theorem 3. *Just 9 of the 27 straight lines on (7) are real. Of the 45 sets of 3 coplanar lines, the lines in 32 sets are not concurrent and those in 13 sets concur.*

5. We consider briefly certain cubic surfaces with singular points. One for which 11 is the only singular point is

$$x^3 + xz^2 + xw^2 + y^2w + yw^2 + xzw = 0. \tag{8}$$

The[7] only real points not on it are 1 and 5. There are only ten lines on this surface, all being real:

$$(a)\ x = y = z,\ (b)\ z = x = y + w,\ (c)\ x = y = z + w,$$
$$(d)\ y = z = x + w,\ (e)\ x = y = 0,\ (f)\ x = w = 0,\ (g)\ x = 0,\ y = w,$$
$$(h)\ w = 0, x = z,\ (i)\ x = w,\ y = z,\ (j)\ x = w = y + z.$$

Of these, f and h alone are in $w = 0$, h counting as a double line of intersection with (8). The only sets of three coplanar lines are a, b, h, in $x = z$, meeting at 11; c, d, h, in $z = x + w$, meeting at 11; e, f, g, in $x = 0$, meeting at 3; f, i, j, in $w = x$, meeting at 8; together with the sets of non-concurring lines a, d, i, in $y = z$; a, c, e, in $x = y$; b, c, j, in $y = z + w$; b, d, g, in $y = x + w$.

Theorem 4. *There are only ten lines on (8) and all are real. Three coplanar lines concur if and only if one of them belongs to the singular pair f, h.*

In (8) we replace x by y, z by $z + y$, y by $x + y + ew$, where $e^2 + e + 1 = 0$; we obtain the real surface

$$yz(z + w) = w(w^2+x^2 + xw), \tag{9}$$

having only ten straight lines of which only two are real. The five real points are those for which $yz = w = 0$, viz., 1, 2, 3, 5, 6.

The only type other than (9) having 1, 2, 3, 5, 6 as its only real points and having a singular point is $w^3 + x^2y + xy^2 + yz^2 + yzw = 0$, containing only three straight lines: $y = w = 0$; $y = w$, $z = x + bw$, $b^2 + b + 1 = 0$; these meet at the only singular point 6.

Another type with only five real points is $xw^2 = y^3 + yz^2 + z^3$, the only lines on which are the six in the planes $x = 0$, $w = 0$.

[1] And on only the surfaces equivalent to (1) or to $xy(x + y) = 0$ under real linear transformation.

[2] They generate all of the 15.6.4.2 real linear automorphs. For, one leaving point 2 fixed must permute the remaining real points 3, 4, 8, 9, 10, 14 in the tangent plane $x = 0$ at 2. These six are permuted transitively by (zw) and γ, both of which leave 2 fixed. An automorph which leaves 2 and 3 fixed, and hence the point 8 collinear with them, must permute 4, 9, 10, 14. These are permuted transitively by β and $\gamma\beta(zw)$, which leave 2 and 3 fixed. An automorph leaving 2, 3 and 4 fixed is the identity or $y^1 = y + x$, which is the transform of β by $(zw)\pi$.

[3] We readily verify that A, B, C and E_{ab}, D_{abc}, with b and c integers and $a^4 = a$, give all of the 27 lines on (1). We have only to consider first the lines whose two equations are solvable for y,z in terms of x,w, and second the lines one of whose equations is $x = 0$ or $w = ax$.

[4] If we make also the like replacement on x,y, we change the left member of (4) into $y(x^2 + xy + y^2)$. The new surface evidently has its seven real points in $y = w$. Replacing w by $w + y$, x by $x + z$ and then z by $z + y$, we get (4).

[5] Its 16 real linear automorphs are generated by the four: $z^1 = z + w$; $x^1 = y + w$, $y^1 = x + w$; $x^1 = y$, $y^1 = x$, $z^1 = x + y + z$; and $y^1 = y + w$, $z^1 = y + z$.

[6] They generate all the twelve real linear automorphs of (7). For, such a transformation must leave fixed the pair of points 9, 12 and hence the tangents $x = y$, $x = 0$ at those points. These points and planes are interchanged by C. If each is fixed, the transformation is

$$x^1 = x,\ y^1 = y,\ z^1 = by + cz + bw,\ w^1 = fy + gz + (f+ 1)w.$$

Since the lines in $x = 0$ are permuted, the case $f = 1$, $g = 0$, is excluded. For $f = g = 0$, we get the identity and A. For $f = 0$, $g = 1$, we get B, BA. For $f = g = 1$, we get AB, BAB.

[7] As shown in the earlier paper, any cubic surface with just 13 real points and having no linear factor is equivalent to (7) or (8).